

BULK SHAPE OF BRANE-WORLD BLACK HOLES

ROBERTO CASADIO

*Dipartimento di Fisica, Università di Bologna and I.N.F.N., Sezione di Bologna, via Irnerio 46,
40126 Bologna, Italy
casadio@bo.infn.it*

LORENZO MAZZACURATI

*Dipartimento di Fisica, Università di Bologna and I.N.F.N., Sezione di Bologna, via Irnerio 46,
40126 Bologna, Italy
mazzacurati@bo.infn.it*

Received (Day Month Year)

Revised (Day Month Year)

We propose a method to extend into the bulk asymptotically flat static spherically symmetric brane-world metrics. We employ the multipole ($1/r$) expansion in order to allow exact integration of the relevant equations along the (fifth) extra coordinate and make contact with the parameterized post-Newtonian formalism. We apply our method to three families of solutions previously appeared as candidates of black holes in the brane world and show that the shape of the horizon is very likely a flat “pancake” for astrophysical sources.

Keywords: black holes, extra dimensions

PACS Nos.: 04.70.-s, 04.70.Bw, 04.50.+h

1. Introduction

In recent years models with extra dimensions have been proposed in which the Standard Model fields are confined to a four-dimensional world viewed as a (infinitely thin) hypersurface (the brane) embedded in the higher-dimensional space-time (the bulk) where (only) gravity can propagate. Of particular interest is the Randall-Sundrum (RS) model with one infinitely extended extra dimension “warped” by a non-vanishing bulk cosmological constant Λ ^{1,2}. In this context, it is natural to study solutions corresponding to compact sources on the brane, such as stars and black holes. This task has proven to be rather complicated and there is little hope to obtain analytic solutions such as those found with one dimension less³. The present literature does in fact provide solutions on the brane^{4,5,6}, perturbative results over the RS background^{7,8} and numerical treatments^{9,10}.

In this letter we investigate how to extend into the bulk asymptotically flat static spherically symmetric solutions on the brane. We hence set the brane cosmological constant to zero by fine tuning Λ to the brane tension, which we denote by 3σ (in

units with $8\pi G_{(5)} = 1$, where $G_{(5)}$ is the five-dimensional gravitational constant), i.e. $\Lambda = -3\sigma^2/2$ ^{1,11}. We then note that the bulk metric in five dimensions must satisfy $(\mu, \nu = 0, \dots, 4)$

$${}^{(5)}R_{\mu\nu} = -\Lambda g_{\mu\nu}. \quad (1.1)$$

On projecting the above equations on the brane and introducing Gaussian normal coordinates x^i ($i = 0, \dots, 3$) and z ($z = 0$ on the brane), one obtains the constraints

$$\left. {}^{(5)}R_{iz} \right|_{z=0} = \left. {}^{(4)}R \right|_{z=0} = 0, \quad (1.2)$$

where ${}^{(4)}R$ is the four-dimensional Ricci scalar and use has been made of the necessary junction equations¹². One can view Eqs. (1.2) as the analogs of the momentum and Hamiltonian constraints in the Arnowitt-Deser-Misner (ADM) decomposition of the metric and their role is to select out admissible field configurations along a given hypersurface of constant z . Such field configurations will then be “propagated” off-brane by the remaining Einstein equations, namely

$${}^{(5)}R_{ij} = -\Lambda g_{ij}. \quad (1.3)$$

The above “Hamiltonian” constraint is a weaker requirement than the purely four-dimensional vacuum equations $R_{ij} = 0$ and is equivalent to $R_{ij} = E_{ij}$, where E_{ij} is (proportional to) the (traceless) projection of the five-dimensional Weyl tensor on the brane¹¹.

In the next Section we outline a general procedure to solve the bulk equations for a given static spherically symmetric brane metric, which we shall then apply to several cases in Section 3. In Section 4 we finally summarize and comment on our results.

2. The procedure

We consider five-dimensional metrics of the form

$$ds^2 = -N(r, z) dt^2 + A(r, z) dr^2 + R^2(r, z) d\Omega^2 + dz^2 \quad (2.1)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ and N , A and R are functions to be determined. For such metrics the momentum constraint is identically solved and the “Hamiltonian” constraint reads

$$2 \left(\frac{N'_B}{N_B} \right)' + \left(\frac{N'_B}{N_B} + \frac{4}{r} \right) \left(\frac{N'_B}{N_B} - \frac{A'_B}{A_B} \right) = \frac{4}{r^2} (A_B - 1), \quad (2.2)$$

where the subscript B is to remind that all functions are evaluated on the brane at $z = 0$, $' \equiv \partial/\partial r$ and we set $R_B = r$ thanks to the spherical symmetry¹³.

Our strategy to construct the bulk metric is then made of three steps:

- (i) choose a metric of the form (2.1) whose projection on the brane,

$$ds|_{z=0}^2 = -N_B(r) dt^2 + A_B(r) dr^2 + r^2 d\Omega^2, \quad (2.3)$$

solves the constraint Eq. (2.2);

- (ii) expand such a metric in powers of $1/r$ (four-dimensional *multipole expansion*) to order n , i.e. introduce the quantities

$$\left. \begin{array}{l} N_n(r, z) \\ A_n(r, z) \\ R_n^2(r, z) \end{array} \right\} \equiv \sum_{k=0}^n \frac{1}{r^k} \left\{ \begin{array}{l} n_k(z) \\ a_k(z) \\ r^2 c_k(z) \end{array} \right\}, \quad (2.4)$$

where $n_k(0)$, $a_k(0)$ and $c_k(0)$ reproduce the particular solution chosen at step 1 (to order n):

$$\sum_{k=0}^n \frac{1}{r^k} \left\{ \begin{array}{l} n_k(0) \\ a_k(0) \\ r^2 c_k(0) \end{array} \right\} = \left\{ \begin{array}{l} N_B(r) \\ A_B(r) \\ r^2 \end{array} \right\} + \mathcal{O}\left(\frac{1}{r^{n+1}}\right); \quad (2.5)$$

- (iii) substitute the sum (2.4) into Eq. (1.3) and integrate exactly in z the (three) equations thus obtained for the functions $n_n(z)$, $a_n(z)$ and $c_n(z)$.

This procedure turns out to be particularly convenient for the problem at hand because it converts Einstein equations (1.3) into three sets of second order ordinary differential equations (in the variable z) of the form

$$\frac{d^2 f_n}{dz^2} - \sigma^2 f_n = F_{k < n}, \quad (2.6)$$

where f_n is any of the functions $n_n(z)$, $a_n(z)$ and $c_n(z)$ ($n \geq 1$), and $F_{k < n}(z)$ a functional of the lower order terms $f_{k < n}$'s and their first and second derivatives. The relevant boundary conditions for Eq. (2.6) are given by the requirement (2.5) for $f_n(0)$ and the junction conditions ¹² which imply $df_n/dz(0) = -\sigma f_n(0)$. We shall not display the (cumbersome) $F_{k < n}$ for simplicity, however they turn out to be such that the Cauchy problem thus defined admits analytic solutions. The entire procedure can then be executed automatically with the aid of an algebraic manipulator to determine the functions f_n recursively, from the lowest order up ^a, with the only limitation of the power and memory of the available computer.

Let us now comment on a few more points regarding the above procedure. First of all, we wish to stress that there is a large freedom in the choice of the metric on the brane. In particular, the coefficients $n_k(0)$ and $a_k(0)$ can be chosen at will, except for the algebraic constraints following from Eq. (2.2). Since such coefficients are related to the shape of the source, this input represents the physical content of the model. A second, related point is the convergence of the series expansion (2.4). It is in general difficult to pinpoint one parameter (among the many possible coefficients of the multipole expansion) whose “smallness” guarantees that orders higher than

^aFor $n = 0$ one has a system of three coupled second order ordinary differential equations for the f_0 's. The corresponding Cauchy problem is solved by the usual warp factor, $f_0 = \exp(-\sigma z)$, which is unique as follows from the usual theorems of uniqueness. We further note that for $r \rightarrow \infty$ our bulk solutions reproduce the RS space-time ¹ as one expects for an asymptotically flat brane

n be negligible. Because of this, we should consider our results as reliable for those values of r and z such that

$$\frac{|f_{n+1}(z)|}{r^{n+1}} \ll \left| \sum_{k=0}^n \frac{f_k(z)}{r^k} \right|, \quad (2.7)$$

for given values of the parameters $n_k(0)$ and $a_k(0)$. In general, for a given z , such a condition will be satisfied for sufficiently large r .

3. Application to brane-world black holes

As examples of brane metrics, we have considered the solutions given in Refs. 4,5,6 which can be expressed in terms of the ADM mass M and the post-Newtonian parameter (PNP) η ¹⁴ measured on the brane. The case with $\eta = 0$ (exact Schwarzschild on the brane) is the well known black string (BS)¹⁵ which extends all along the extra dimension. The BS is known to suffer of serious stability problems^{15,16}, e.g. the Kretschmann scalar,

$$K^2 \equiv {}^5R_{\mu\nu\lambda\rho} {}^5R^{\mu\nu\lambda\rho} = \frac{5}{2} \sigma^4 + 48 M^2 \frac{e^{4\sigma z}}{r^6}, \quad (3.1)$$

diverges on the AdS horizon ($z \rightarrow \infty$). One is therefore led to conclude that black holes on the brane must depart from Schwarzschild and have $\eta \neq 0$. As was suggested in⁶, the interesting cases are those with $\eta < 0$, since $\eta > 0$ implies some sort of anti-gravity effects (see later for further comments).

Short distance tests of Newtonian gravity yield the bound $\sigma^{-1} < 1 \text{ mm}^{-1}$ and $|\eta| < 10^{-3}$ from solar system tests¹⁴. Since we want to study astrophysical sources of solar mass size, in the following we shall often refer to the typical values

$$M = 10^7 \sigma^{-1} \sim 1 \text{ km}, \quad \eta = -10^{-4}. \quad (3.2)$$

In such a range ($M\sigma \gg 1$ and $|\eta| \ll 1$) one finds a qualitatively identical behavior for all brane metrics in Refs. 4,5,6, so we shall just give the results for case I of Ref. 6 (see also⁵), that is

$$N_B = 1 - \frac{2M}{r} \\ A_B = \frac{\left(1 - \frac{3M}{2r}\right)}{\left(1 - \frac{2M}{r}\right) \left[1 - \frac{3M}{2r} \left(1 + \frac{4}{9}\eta\right)\right]}, \quad (3.3)$$

where $r = r_h \equiv 2M$ is the event horizon and the remaining (non-vanishing) PNPs are $\beta = \gamma = 1 + \frac{1}{3}\eta$.

We applied the above procedure to the brane metric (3.3) and were able to solve

the corresponding Eqs. (2.6) up to $n = 19$. For brevity, we just display a few terms:

$$\begin{aligned} N_5 &= e^{-\sigma z} \left[1 - \frac{2M}{r} - \frac{\eta M^2 (1 - e^{\sigma z})^2}{3r^4} \left(1 + \frac{M}{r} \right) \right] \\ A_3 &= e^{-\sigma z} \left[1 + \frac{2M}{3r} (3 + \eta) + \frac{4M^2}{r^2} \left(1 + \frac{\eta}{12} + \frac{\eta^2}{9} \right) \right. \\ &\quad \left. + \frac{8M^3}{r^3} - \frac{2\eta M}{3r^3} \frac{(1 - e^{\sigma z})^2}{\sigma^2} \right] \\ \frac{R_5^2}{r^2} &= e^{-\sigma z} \left[1 + \frac{\eta M}{3r^3} \frac{(1 - e^{\sigma z})^2}{\sigma^2} \left(1 + \frac{2M}{r} + \frac{7M^2}{2r^2} \right) \right] \end{aligned} \quad (3.4)$$

It is important to note the appearance of positive exponentials in the metric functions. Such terms (which also show up at higher orders) are non-perturbative in z , which makes the expansion in $1/r$ preferable (or at least complementary) to the expansion for small z .

For $\eta < 0$ one generally finds that, for every $n \geq 3$ and $r > 0$, there exists a corresponding value of z [say $z_n^{\text{axis}}(r)$] such that $R_n^2(r, z_n^{\text{axis}}) = 0$. Since $4\pi R^2$ is the proper area of the sphere $t = r = z = \text{constant}$, this seems to indicate that the axis of cylindrical symmetry is given, in our Gaussian reference frame, by a line $z = z^{\text{axis}}(r)$ (which should be exactly obtained in the limit $n \rightarrow \infty$). Since R_n^2 is determined up to corrections of order $n + 1$, which in general do not vanish for $z = z_n^{\text{axis}}(r)$, one cannot consider this as a mathematically rigorous proof [the condition (2.7) obviously fails for $z = z_n^{\text{axis}}$]. However, we found that in the physically interesting range of the parameter space (M, η) the $1/r$ expansion yields rather stable values of $z_n^{\text{axis}}(r)$ in a wide span of n . The stability improves for larger values of r [as could be inferred from (2.7)] and becomes very satisfactory for $r \gtrsim r_h$ (see Fig. 1). From Eq. (3.4) and $M\sigma \gg 1$, $r \gg r_h$, one finds

$$z_5^{\text{axis}} \sim \frac{1}{2\sigma} \ln \left(\frac{3\sigma^2 r^3}{-\eta M} \right), \quad (3.5)$$

which numerically agrees fairly well with $z_{19}^{\text{axis}}(r)$. Further, we have checked that $R_n(r_1, z) > R_n(r_2, z)$ for $r_1 > r_2$ and $0 \leq z < z_n^{\text{axis}}(r_2)$, so that lines of constant r do not cross and the Gaussian normal form (2.1) of the metric is preserved for $\eta < 0$ in the bulk within our approximation (for an example see Fig. 2).

It is interesting to compare our solutions for $\eta < 0$ with the BS¹⁵. In particular, one would like to see the shape of the horizon in the bulk, knowing that for the BS it does not close but extends all the way to the AdS horizon. First we note that, if the horizon closes in the bulk, then it must cross the axis of cylindrical symmetry at a point (the “tip”) of finite coordinates $(r^{\text{tip}}, z^{\text{tip}})$ where $N = R^2 = 0$. Of course, we just have such equations explicitly at order n ,

$$N_n(r_n^{\text{tip}}, z_n^{\text{tip}}) = R_n^2(r_n^{\text{tip}}, z_n^{\text{tip}}) = 0. \quad (3.6)$$

6 R. Casadio and L. Mazzacurati

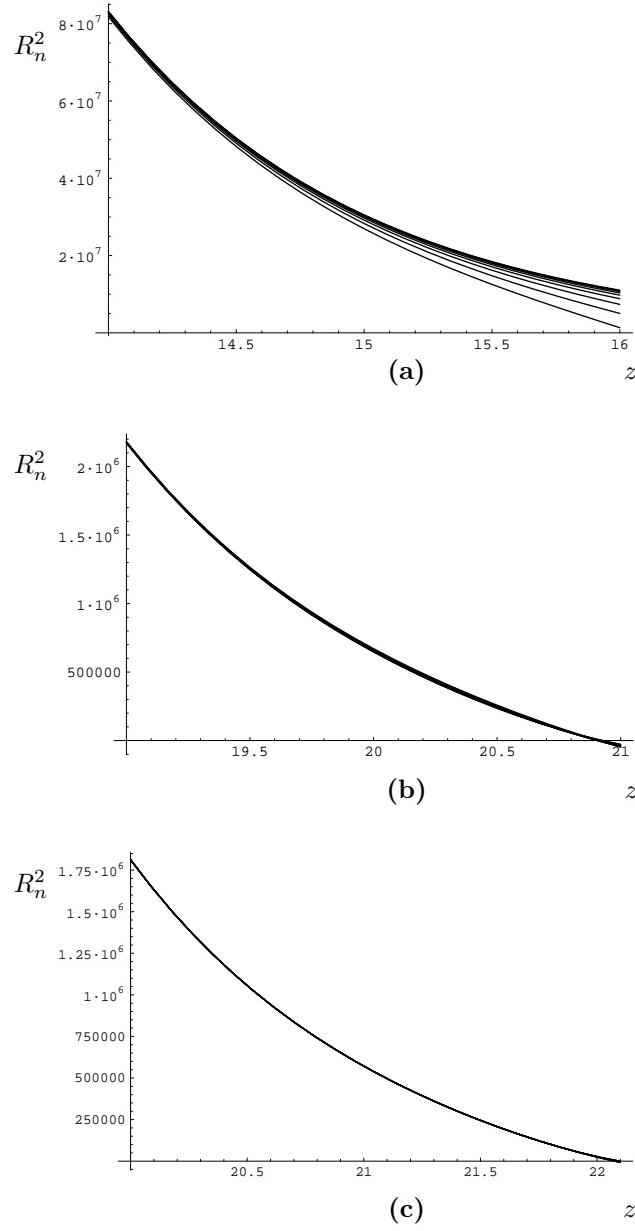


Fig. 1. The function R_n^2 near z^{axis} evaluated to orders $n = 10 \rightarrow 19$ for the data (3.2) and $r = M$ (a), $r = r_h = 2M$ (b), and $r = 3M$ (c) (lengths are in units of σ^{-1}).

For large values of n , one can solve Eqs. (3.6) numerically and find the “tip”. More

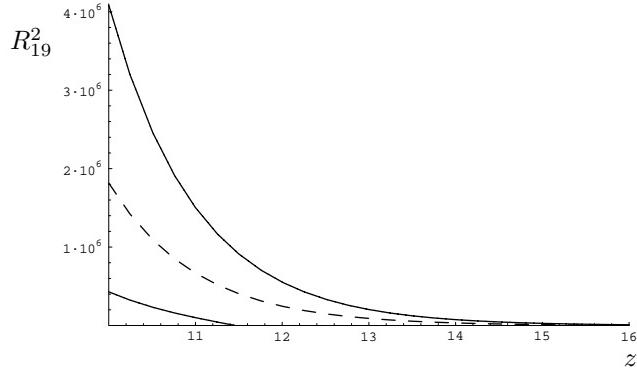


Fig. 2. The function R^2 evaluated to order $n = 19$ for $r = M$, $r_h = 2M$ (dashed line) and $10M$ with $M = 10^6 \sigma^{-1}$ and $\eta = -10^{-4}$.

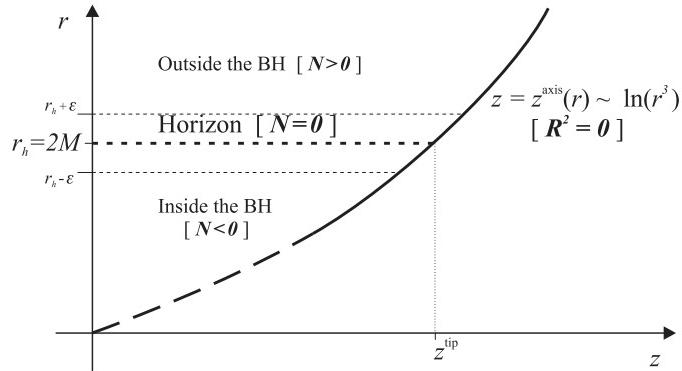


Fig. 3. Qualitative picture of the bulk structure.

in detail, for $n = 19$ one has

$$N_{19}(r_h + \varepsilon, z) > 0 \quad \text{and} \quad N_{19}(r_h - \varepsilon, z) < 0 , \quad (3.7)$$

for $\varepsilon \gtrsim (3/100) r_h$ and $0 < z \lesssim 0.99 z_{19}^{\text{axis}}(r)$. Thus, to a very good approximation, $N_n(r_h, z) \simeq 0$ for $0 \leq z < z_n^{\text{axis}}(r_h)$ when $M\sigma \gg 1$ and $|\eta| \ll 1$ and negative. A good parameterization for the horizon is thus given by $r \simeq r_h$ and $0 \leq z \lesssim z_n^{\text{axis}}(r_h) \simeq z_{19}^{\text{tip}}$ (see Fig. 3). For the typical parameters (3.2) we have $r_{19}^{\text{tip}} \simeq 0.97 r_h$ and $z_{19}^{\text{tip}} \simeq 20.8 \sigma^{-1}$ which is very close to $z_{19}^{\text{axis}}(r_h) = 20.9 \sigma^{-1}$. This all strongly suggests that the horizon does close in the bulk, as previously obtained by numerical analysis^{9,10} (for a comparison with the BS see Fig. 4).

One can now get an estimate of how flattened the horizon is towards the brane by comparing the proper length of a circle on the horizon which lies entirely on the brane, $C_{||} = 2\pi r_h \simeq 1.3 \cdot 10^8 \sigma^{-1}$, with the length of an analogous curve perpendicular to the brane, $C_{\perp} \simeq 4z_{19}^{\text{tip}} \simeq 84 \sigma^{-1}$. Since their ratio is huge, one can

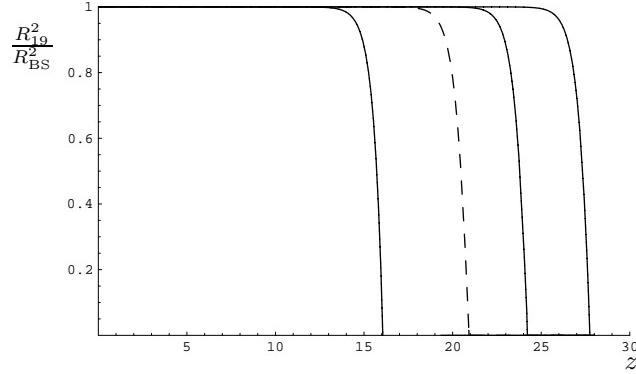


Fig. 4. The function R^2/R_{BS}^2 evaluated to order $n = 19$ for $r = M$, $r_h = 2M$ (dashed line), $10M$ and $100M$ with $M = 10^7 \sigma^{-1}$ and $\eta = -10^{-4}$.

in fact speak of a “pancake” horizon as was suggested, e.g., in Ref. 7.

It is interesting to note that for $n = 5$, one can still solve Eqs. (3.6) analytically and finds

$$z_5^{\text{tip}} = \frac{1}{\sigma} \ln \left[1 + \frac{a r_h \sigma}{\sqrt{-\eta}} \right] \sim \frac{1}{\sigma} \ln \left(\frac{M \sigma}{\sqrt{-\eta}} \right) , \quad (3.8)$$

where $a \simeq 0.5$ and we used $M \sigma \gg 1$ in the final expression. This yields $z_5^{\text{tip}} = 20.7 \sigma^{-1}$ for the parameters (3.2), in excellent agreement with the numerical value obtained at order 19. In light of this stability, one can therefore approximate the dependence of the exact z^{tip} on the black hole ADM mass M from Eq. (3.8) and obtains that the area of the (bulk) horizon is approximately equal to the four-dimensional (brane) expression ^b,

$${}^{(5)}\mathcal{A} = 4\pi \int_0^{z_5^{\text{tip}}} R^2(r_h, z) dz \simeq \frac{16\pi}{\sigma} (2M)^2 , \quad (3.9)$$

where we again used $M \sigma \gg 1$. Eqs. (3.8) and (3.9) again supports the idea of a “pancake” shape for the horizon (see also 7 for the logarithmic dependence of z^{tip} on M).

Drawing upon the above picture, in particular the crossing of lines of constant r with the axis of cylindrical symmetry at finite z , one can infer that the space-times we obtain do not suffer of one of the instabilities of the BS, namely the diverging Kretschmann scalar ¹⁵. In fact, K^2 is still an increasing function of z along lines of

^bThe fundamental (possibly TeV scale) five-dimensional gravitational coupling $G_{(5)} \sim G_N/\sigma$, where G_N is the four-dimensional Newton constant ¹. Thus, from (3.9) one has ${}^{(5)}\mathcal{A}/G_{(5)} \sim M^2/G_N \sim {}^{(4)}\mathcal{A}/G_N$.

constant r , but one has

$$K^2 - \frac{5}{2} \sigma^4 = \bar{a} \frac{e^{4\sigma z}}{r^6} + \mathcal{O}\left(\frac{1}{r^7}\right) \leq \bar{b} \sigma^4 + \mathcal{O}\left(\frac{1}{r}\right), \quad (3.10)$$

where we used Eq. (3.5) to maximize K^2 uniformly in z . The coefficients \bar{a} and \bar{b} depend on the parameters of the multipole expansion, correctly vanish in pure RS and $\bar{b} = 48 M^2$ for $\eta = 0$ [the BS, cfr. Eq. (3.1)]. The remaining problem of stability under (linear) perturbations¹⁶ is a difficult one to tackle and we do not attempt at it here.

As we mentioned previously, the cases with $\eta > 0$ show a very different qualitative behavior. One in fact finds that $R_n^2(r, z)$ is generically a (monotonically) increasing function of z for all (sufficiently large) values of r , as one would indeed expect on a negative tension brane¹¹. However, for any $r_1, r_2 > 0$ there now exists $z_n^*(r_1, r_2)$ such that $R_n^2(r_1, z_n^*) = R_n^2(r_2, z_n^*)$, i.e. space-like geodesics of constant r display caustics and the Gaussian coordinates (r, z) do not cover the whole bulk¹³.

4. Conclusions and outlook

We have explained in some detail a method to extend into the bulk a given asymptotically flat static spherically symmetric metric on the brane which is based on the multipole ($1/r$) expansion. The application of our method to candidate metrics^{4,5,6} for astrophysical sources led us to conclude that black hole horizon closes in the bulk and indeed has the shape of a “pancake”. Our solutions depend on three parameters (σ, M and η), although one could argue that only two of them are independent. For instance, in order to recover the four-dimensional Schwarzschild metric when the extra dimension shrinks to zero size ($\sigma \rightarrow \infty$ in the brane equations¹¹), one can guess $|\eta| \sim 1/M\sigma$. Reducing the number of dimensions is however a singular limit of the five-dimensional metric, so it is critical to obtain precise relations among the parameters by this procedure. It is also difficult to extract sensible results for tiny black holes ($M\sigma \ll 1$), e.g. from Eq. (3.8) one now gets $z_5^{\text{tip}} \sim M$ and Eq. (3.9) yields the relation for five-dimensional Schwarzschild black holes ${}^{(5)}\mathcal{A} \sim M^3$ ¹⁷. Our expansion however suggests that the horizon departs significantly from the line $r = r_h$ when $M\sigma \ll 1$ and the above estimate is very rough. The dependence of the horizon area on the ADM mass is crucial to study the Hawking evaporation and we hope to return to it in the future.

Acknowledgments

We thank A. Fabbri for contributing to the early part of the work. R. C. thanks C. Germani and R. Maartens for comments and suggestions.

References

1. L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999); Phys. Rev. Lett. **83**, 4690 (1999).

10 *R. Casadio and L. Mazzacurati*

2. N. Arkani-Hamed, S. Dimopoulos, G.R. Dvali and N. Kaloper, Phys. Rev. Lett. **84**, 586 (2000).
3. R. Emparan, G.T. Horowitz and R.C. Myers, JHEP **0001**, 007 (2000)
4. N. Dadhich, R. Maartens, P. Papadopoulos and V. Rezania, Phys. Lett. **B 487**, 1 (2000).
5. C. Germani and R. Maartens, Phys. Rev. D **64**, 124010 (2001).
6. R. Casadio, A. Fabbri and L. Mazzacurati, Phys. Rev. D **65**, 084040 (2002)
7. S.B. Giddings, E. Katz and L. Randall, JHEP **0003**, 023 (2000)
8. J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (2000)
9. A. Chamblin, H.S. Reall, H. Shinkai and T. Shiromizu, Phys. Rev. D **63**, 064015 (2001)
10. T. Wiseman, Phys. Rev. D **65**, 124007 (2002).
11. T. Shiromizu, K. Maeda and M. Sasaki, Phys. Rev. D **62**, 024012 (2000).
12. W. Israel, Nuovo Cimento **B 44**, 1 (1966); **B 48**, 463 (1966).
13. R. Wald, *General Relativity*, (Chicago University Press, Chicago, 1984).
14. C.M. Will, *Theory and experiment in gravitational physics*, 2nd ed. (Cambridge University Press, Cambridge, 1993); Living Rev. Rel. **4**, 4 (2001).
15. I. Chamblin, S. Hawking and H.S. Reall, Phys. Rev. D **61**, 065007 (2000).
16. R. Gregory, Class. Quant. Grav. **17**, L125 (2000).
17. R.C. Myers and M.J. Perry, Ann. Phys. **172**, 304 (1986).